Describing a Situation of Strategic Interaction

- We need to know four things:
  (i) *The players:* Who is involved?
  (ii) *The rules:* Who moves when? What do they know when they move? What can they do?
  (iii) *The outcomes:* For each possible set of actions by the players, what is the outcome of the game?
  (iv) *The payoffs:* What are the players' preferences (i.e., utility functions) over the possible outcomes?
- *Strategic interdependence*: Each individual's welfare depends not only on her own actions but also on the actions of the other individuals.
  - Moreover, the actions that are best for her to take may depend on what she expects the other players to do.

Maching Pennies

(i) There are two players, denoted 1 and 2.

(ii) Each player simultaneously puts a penny down, either heads up or tails up.

(iii) If the two pennies match (either both heads up or both tails up), player 1 pays 1 dollar to player 2; otherwise, player 2 pays 1 dollar to player 1.

Meeting in New York

(i) Two players, Mr. Thomas and Mr. Schelling.

(ii) The two players are separated and cannot communicate. They are supposed to meet in New York City at noon for lunch but have forgotten to specify where. Each must decide where to go (each can make only one choice).

(iii) If they meet each other, they get to enjoy each other's company at lunch. Otherwise, they must eat alone.

(iv) They each attach a monetary value of 100 dollars to the other's company (their payoffs are each 100 dollars if they meet, 0 dollars if they do not).

- Even the task of coordination can have a strategic nature.
  - Each player's payoff depends on what the other player does; and more importantly, *each player's optimal action depends on what he thinks the other will do*.

Common Knowledge

- It is a basic postulate of game theory that all players know the structure of the game, know that their rivals know it, know that their rivals know that they know it, and so on. In theoretical parlance, we say that the structure of the game is *common knowledge* [see Aumann (1976) and Milgrom (1981) for discussions of this concept].

Extensive Form Representation

- $\Gamma_E = [\mathscr{X}, \mathscr{A}, I, p(\cdot), \alpha(\cdot), \mathscr{H}, H(\cdot), \iota(\cdot), \rho(\cdot), u]$
  - $\mathscr{X}$: *Nodes*
  - $\mathscr{A}$: Possible *actions*
  - $\{1, \ldots, I\}$: *Players*
  - $p : \mathscr{X} \to \{\mathscr{X} \cup \emptyset\}$
    - $\exists x_0$ (*initial node* (or *root*)): $p(x_0) = \emptyset$, $p(x) \neq \emptyset \; \forall x \neq x_0$
      - $s(x) = p^{-1}(x)$
      - *Terminal nodes*: $T = \{x \in \mathscr{X} : s(x) = \emptyset\}$
      - *Decision nodes*: $D = \mathscr{X} \setminus T$
    - $P(x) \cap S(x) = \emptyset \; \forall x$
      - $P(x) = \{x' \in \mathscr{X} : x' = x^1,\, x^1 = p(x^2), \ldots, x^{K-1} = p(x^K)\; x^K = x$ for some $x^1, \ldots, x^K\}$ (*predecessors* of x)
      - $S(x) = \{x' \in \mathscr{X} : x = x^1,\, x^2 \in s(x^1), \ldots, x^K \in s(x^{K-1}),\, x^K = x'$

for some $x^1, \ldots, x^K\}$ (*successors* of x)

- $\alpha : \mathscr{X} \setminus \{x_0\} \to \mathscr{A}$
  - $x', x'' \in s(x)$ and $x' \neq x'' \Rightarrow \alpha(x') \neq \alpha(x'')$
- Partition $\mathscr{H}$ of $\mathscr{X}$: *Information sets*
  - $H : \mathscr{X} \to \mathscr{H}$, $x \in H(x)$
  - $c(x) = c(x')$ if $H(x) = H(x')$
    - $c(x) = \{a \in \mathscr{A} : a = \alpha(x')$ for some $x' \in s(x)\}$
- $\iota : \mathscr{H} \to \{0, 1, \ldots, I\}$
  - $\mathscr{H}_i = \{H \in \mathscr{H} : i = \iota(H)\}$
- $\rho : \mathscr{H}_0 \times \mathscr{A} \to [0,1]$
  - $\rho(H,a) = 0$ if $a \notin C(H)$ and $\sum_{a \in C(H)} \rho(H,a) = 1$ for all $H \in \mathscr{H}_0$.
    - $C(H) = \{a \in \mathscr{A} : a \in c(x)$ for $x \in H\}$
- $u = \{u_i(\cdot)\}_{i \in \{1,\ldots,I\}}$, $u_i : T \to \mathbb{R}$ (*payoff* function)
  - Bernoulli utility function for a random realization of outcomes.

- $[\mathscr{X}, \mathscr{A}, I, p(\cdot), \alpha(\cdot), \mathscr{H}, H(\cdot), \iota(\cdot), \rho(\cdot)]$ is formally known as an extensive *game form*, adding u leads to a *game* represented in extensive form.
  - See Kuhn (1953) or Section 2 or Kreps and Wilson (1982) for additional discussion of this and other points regarding the extensive form.
- Games with a finite number of nodes are known as *finite games*.

Perfect Recall

- If the following two conditions hold:

  (i) $H(x) = H(x') \Rightarrow$ Neither $x \in P(x')$ nor $x \in S(x')$.

  (ii) $x, x' \in D$, $H(x) = H(x')$, $x'' \in P(x)$, $\iota(H(x'')) = \iota(H(x))$
  $\Rightarrow \exists x'''$: $x''' \in P(x') \cap H(x'')$, $\alpha(s(x''') \cap P(x')) = \alpha(s(x'') \cap P(x))$

- A player does not forget what she once knew, including her own actions.

- All the games we consider in this book satisfy the property of perfect recall.

Perfect Information

- If $|H| = 1\ \forall H \in \mathscr{H}$.
  - *Imperfect information* otherwise.
- When it is a player's turn to move, she is able to observe all her rival's previous moves (including *random moves of nature*).
  - The concept of an *informaion set* allows us to accommodate the possibility that this is not so.
    - When play has reached one of the decision nodes in the information set and it is that player's turn to move, she does not know which of these nodes she is actually at. The reason for this ignorance is that the player does not observe something about what has previously transpired in the game.
    - The use of information sets also allows us to capture play that is simultaneous rather than sequential.

Strategy

- $s_i : \mathscr{H}_i \to \mathscr{A}$ such that $s_i(H) \in C(H)$ for all $H \in \mathscr{H}_i$.
- *Complete contingent plan*, or *decision rule*, that specifies how the player will act in *every possible distinguishable circumstance* in which she might be called upon to move.
  - The set of such circumstances is represented by her collection of information sets, with each information set representing a different distinguishable circumstance in which she may need to move.

Normal (or Strategic) Form

- $\Gamma_N = [I, \{S_i\}, \{u_i(\cdot)\}]$
  - S_i: i's (*pure*) strategy set
  - $u_i : S \to \mathbb{R}$
    - $S = S_1 \times \cdots \times S_I$
- Is the scenario in which players simultaneously write down their strategies and submit them to a referee really equivalent to their playing the game over time as described in the extensive form?
  - For the simultaneous-move games, the normal form captures all the strategically relevant information.

Mixed Strategy

- $\sigma_i : S_i \to [0,1]$, $\sigma_i(s_i) \geq 0\ \forall s_i \in S_i$, $\sum_{s_i \in S_i} \sigma_i(s_i) = 1$
- The von Neumann-Morgenstern utility:

$$u_i(\sigma) = \sum_{s \in S} \sigma_1(s_1)\sigma_2(s_2)\cdots\sigma_I(s_I)u_i(s)$$

  - $\sigma = (\sigma_1, \ldots, \sigma_I)$
- *Mixed extention* of $S_i = \{s_{1i}, \ldots, s_{Mi}\}$:

$$\Delta(S_i) = \{(\sigma_{1i}, \ldots, \sigma_{Mi}) \in \mathbb{R}^M : \sigma_{mi} \geq 0 \forall m = 1, \ldots, M \text{ and } \sum_{m=1}^{M} \sigma_{mi} = 1\}$$

- Alternative interpretation: Each player i has access to a private signal θ_i that is uniformly distributed on the interval $[0,1]$ and is independent of other players' signals, and specifies a pure strategy $s_i(\theta_i) \in S_i$ for each realization of θ_i.

Behavior strategy

- $\lambda_i : \{(a,H) : H \in \mathscr{H}_i, a \in C(H)\} \to [0,1]$
  - $\lambda_i(a,H) \geq 0, \sum_{a \in C(H)} \lambda_i(a,H) = 1.$
- Rather than randomizing over the potentially very large set of pure strategies in S_i, she could randomize separately over the possible actions at each of her information sets $H \in \mathscr{H}_i$.
- For games of perfect recall: For any behavior strategy of player i, there is a mixed strategy for that player that yields exactly the same distribution over outcomes for any strategies, mixed or behavior, that might be played by i's rivals, and vice versa [Kuhn (1953)].

Exercises

- **7.C.1**$^{\text{A}}$ Suppose that in the Meeting in New York game, there are two possible places where the two players can meet: Grand Central Station and the Empire State Building. Draw an extensive form representation (game tree) for this game.

- **7.D.2**$^{\text{A}}$ Depict the normal forms for Matching Pennies Version C and the standard version of Matching Pennies.
  - Matching Pennies Version C is just like Matching Pennies Version B except that when player I puts her penny down, she keeps it covered with her hand, Hence, player 2 cannot see player 1's choice until after player 2 has moved.
    - Matching Pennies Version B is identical to Matching Pennies except that the two players move sequentially, rather than simultaneously. In particular, player I puts her penny down (heads up or

tails up) first. Then, after seeing player 1's choice, player 2 puts her penny down.

- **7.E.1**[B] Consider the two-player game: whose extensive form representation (excluding payoffs) is depicted in Figure 7.Ex.1.

  (b) Show that for any behavior strategy that player 1 might play, there is a realization equivalent mixed strategy; that is, a mixed strategy that generates the same probability distribution over the terminal nodes for any mixed strategy choice by player 2.

  (c) Show that the converse is also true: For any mixed strategy that player I might play, there is a realization equivalent behavior strategy.

  (d) Suppose that we change the game by merging the information sets at player 1's second round of moves (so that all four nodes are now in a single information set). Argue that the game is no longer one of perfect recall. Which of the two results in (b) and (c) still holds?

References

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