Infinitely Repeated Games

- An infinitely repeated game consists of an infinite sequence of repetitions of a one-period simultaneous-move game, the *stage game*.
  - In the stage game, each player i has a strategy set S_i; $q_i \in S_i$ is a particular feasible action for player i.
    - Player i's payoff function: $\pi_i(q)$
      - $q = (q_1, \ldots, q_I)$
    - One-period best-response payoff: $\hat{\pi}_i(q_{-i}) = \max_{q'_i \in S_i} \pi_i(q'_i, q_{-i})$.
- The players discount payoffs with discount factor $\delta \in (0, 1)$.
- Players observe each other's action choices in each period (and have perfect recall).
  - A pure strategy for player i, s_i, is a sequence of functions $\{s_{it}(\cdot)\}_{t=1}^{\infty}$, mapping from the history of previous action choices (denoted H_{t-1}) to his action choice in period t, $s_{it}(H_{t-1}) \in S_i$.

- *Outcome path* $Q(s)$: an infinite sequence of actions $\{q_t\}_{t=1}^{\infty}$ that will actually be played when the players follow strategies s.
  - Discounted payoff from outcome path Q: $v_i(Q) = \sum_{t=0}^{\infty} \delta^t \pi_i(q_{t+1})$.
  - *Average payoff* from outcome path Q: $(1-\delta)v_i(Q)$
  - The discounted continuation payoff from outcome path Q from some period t onward (discounted to period t): $v_i(Q,t) = \sum_{\tau=0}^{\infty} \delta^{\tau} \pi_i(q_{t+\tau})$.

Nash Reversion

- The strategies that call for each player i to play his stage game Nash equilibrium action in every period, regardless of the prior history of play, constitute an SPNE for *any* value of $\delta < 1$.
- A strategy profite in an infinitely repeated game is one of *Nash reversion* if each ptayer's strategy calls for playing some outcome path until someone defects and playing a stage game Nash equilibrium q^* thereafter.
- A Nash reversion strategy profile that calls for playing Q prior to any deviation is an SPNE $\iff \hat{\pi}_i(q_{-it}) + \frac{\delta}{1-\delta}\pi_i(q^*) \leq v_i(Q,t)\ \forall t\ \forall i$.
- Consider a two player case with $S_i \subset \mathbb{R}\ \forall i$. Suppose also that $\pi_i(q)$ is differentiable at a stage game Nash equilibrium q^*, with $\partial \pi_i(q^*)/\partial q_j \neq 0\ \forall j \neq i\ \forall i$. Then, there is some q' with $\pi(q) \gg \pi(q^*)$ whose infinite repetition is the outcome path of an SPNE that uses Nash reversion.
- Outcome path Q can be sustained as an SPNE outcome path using Nash

reversion $\Rightarrow$ it can be so sustained for any $\delta' \geq \delta$.

- *Nash reversion folk theorem* [Friedman (1971)]: $\pi_i(q) > \pi_i(q^*)$ $\Rightarrow \exists \underline{\delta} \; \forall \delta > \underline{\delta}$: infinite repetition of q is the outcome path of an SPNE using Nash reversion strategies.
- *Minimax payoff*: $\underline{\pi}_i = \min_{q_{-i}} \max_{q_i} \pi_i(q)$.
  - Regardless of the strategies played by his rival, player i's average payoff in the infinitely repeated game or in any subgame within it cannot be below $\underline{\pi}_i$.
    - Payoffs that strictly exceed $\underline{\pi}_i$ for each player i are known as *individually rational payoffs*.
- Consider a two player case with $S_i \subset \mathbb{R} \; \forall i$. Suppose also that $\pi_i(q)$ is differentiable at a stage game Nash equilibrium q^*, with $\partial \pi_i(q^*)/\partial q_j \neq 0 \; \forall j \neq i \; \forall i$. Then: $\pi(q^*) > \underline{\pi}_i \; \forall i \Rightarrow$ there is some SPNE with discounted payoffs to the players v' such that $(1-\delta)v'_i < \pi_i(q^*) \; \forall i$.

Folk Theorem

- We focus on the case with two players and pure strategies.
  - See Fudenberg and Maskin (1986) and Fudenberg and Maskin (1991).
    * With more than two players, the result requires that the set of feasible payoffs satisfy an additional "dimensionality" condition.
- *Folk Theorem*: $\pi \gg \underline{\pi} \Rightarrow \exists \underline{\delta} < 1 \ \forall \delta > \underline{\delta}$: π are the average payoffs arising in an SPNE.
  - *One deviation principle*: If no single-period deviation followed by conformity with the stategies is worthwhile, then neither is any multiperiod deviation (this is a general principle of dynamic programming).
- The theorem's name refers to the fact that some version of the result was known in game theory "folk wisdom" well before its formal appearance in the literature.

- The original appearances of the result in the literature actually analyzed infinitely repeated games *without* discounting [see, for example, Rubinstein (1979)].
- For arbitrary δ, constructing the full set of SPNEs is a delicate process. Each SPNE, whether collusive or punishing, uses other SPNEs as threatened punishments. For details on how this is done, see the original contributions by, e.g., Abreu (1986) and Abreu (1988).

Exercises

- **12.D.3**$^{\mathbf{B}}$ Consider an infinitely repeated Cournot duopoly with discount factor $\delta < 1$, unit costs of $c > 0$, and inverse demand function $p(q) = a - bq$, with $a > c$ and $b > 0$.

  (a) Under what conditions can the symmetric joint monopoly outputs $q_1 = q_2 = q^m/2$ be sustaincd with strategies that call for $(q^m/2, q^m/2)$ to be played if no one has yet deviated and for the single-period Cournot (Nash) equilibrium to be played otherwise?

  (b) Derive the minimal level of δ such that output levels $(q_1, q_2) = (q, q)$ with $q \in [(((a-c)/(2b)), ((a-c)/b)]$ are sustainable through Nash reversion strategies. Show that this level of δ, $\delta(q)$, is an increasing, differentiable function of q.

References

Abreu, Dilip (1986) "Extremal equilibria of oligopolistic supergames," *Journal of Economic Theory*, Vol. 39, No. 1, pp. 191–225.

——— (1988) "On the theory of infinitely repeated games with discounting," *Econometrica*, pp. 383–396.

Friedman, James W (1971) "A non-cooperative equilibrium for supergames," *The Review of Economic Studies*, pp. 1–12.

Fudenberg, Drew and Eric Maskin (1986) "The folk theorem in repeated games with discounting or with incomplete information," *Econometrica*, pp. 533–554.

——— (1991) "On the dispensability of public randomization in discounted repeated games," *Journal of Economic Theory*, Vol. 53, No. 2, pp. 428–438.

Rubinstein, Ariel (1979) "Equilibrium in supergames with the overtaking criterion," *Journal of Economic Theory*, Vol. 21, No. 1, pp. 1–9.